FUNCTION ALGEBRAS ON A MANIFOLD

by Michael Freeman

This note is a brief exposition of recent results. No proofs are offered, except for some very simple examples.

Let M be a compact Hausdorff space and $C(M)$ the algebra of all continuous complex valued functions on M. $C(M)$ is a Banach algebra under the uniform norm

$$\|f\| = \sup_{p \in M} |f(p)|.$$

A *function algebra* on M is a closed subalgebra A of $C(M)$ containing the identity and separating the points of M. The latter condition means that if p and q are distinct points of M there exists a function f in A such that $f(p) \neq f(q)$.

A classical example is obtained when M is a compact set in the plane and A is the function algebra generated by z, the usual complex coordinate. Here the description of A is the problem of uniform approximation on M by polynomials. If M does not separate the plane, then Mergelyan [5] has shown that A is the algebra of all continuous functions on M which are holomorphic at interior points. If we replace z by a more general continuous function f on M, then f must be injective in order for the algebra A it generates to separate the points of M. Thus f is a homeomorphism, and it transports the new problem back to the previous one with M replaced by $f(M)$. Since $f(M)$ does not separate the plane if this is true of M, f induces an isometric isomorphism of A with the algebra of continuous functions on $f(M)$ which are holomorphic at interior points.

It should be mentioned that M does not separate the plane if and only if the maximal ideal space M_A of the Banach algebra A contains no other complex homomorphisms than those arising from evaluation at points of M. When this occurs for a function algebra A we say $M_A = M$.

In fact, the condition $M_A = M$ is usually assumed for the results below, and we should review some basic facts about M_A. We shall discuss algebras A generated by a set F of continuous complex valued functions separating the points of M. Thus A consists of all uniform limits of polynomials in the functions of F. In case $F = \{f_1, \cdots, f_n\}$ is finite, the functions f_j are the coordinates of a homeomorphism also denoted $F: M \to \mathbf{C}^n$. By means of this map the description of A becomes equivalent to the problem of characterizing those continuous functions on $F(M)$ which can be uniformly approximated by polynomials in $z_1, \cdots, z_n$, the usual coordinates in $\mathbf{C}^n$. The condition that $M_A = M$ is equivalent [7] to requiring that $F(M)$ be *polynomially convex*. A compact set K in $\mathbf{C}^n$ enjoys this property if for each point $z \notin K$ there exists a polynomial p in n variables such that

$$|p(z)| > \sup_{w \in K} |p(w)|.$$

An application of the maximum principle shows that when $n = 1$ this reduces to Mergelyan's condition that $\mathbf{C} - K$ be connected. However, a useful characterization of this property is presently unavailable in higher dimensions. In general, M_A can be identified [7] with the smallest polynomially convex set containing $F(M)$, called its polynomial hull.

Returning to compact sets M in the plane, let us consider the algebra A generated by a set $F = \{f, g\}$ of two functions. We suppose that M is a closed disk centered at 0. This problem was also treated by Mergelyan [5], who found as a consequence of his result above that if $g = z$ and f is a real valued function none of whose level sets separate the plane, then A consists of all continuous functions on M which are holomorphic at the interior points of each level set of f. In particular, if no level set has interior points, then $A = C(M)$.

A different tack was taken by Wermer [8] who considered z and a continuously differentiable complex valued function f. He showed that the set

$$E = \{\zeta \in M: \bar{\partial} f(\zeta) = 0\}$$

plays an important role in the structure of A ($\bar{\partial} f(\zeta) = 0$ means that f satisfies the Cauchy-Riemann equations at ζ). In fact, under the hypothesis that $M_A = M$, he found that A is the set of all continuous functions f on M whose restriction $f \mid E$ to E is in the uniform closure $R(E)$ on E of the rational functions with poles outside E. Since much is known about the algebra $R(E)$ this amounts to a very explicit description of A. For example, if E has plane measure zero, then a theorem of Hartogs and Rosenthal [3] yields $R(E) = C(E)$, so that $A = C(M)$. If E has only finitely many complementary components in the plane, then $R(E)$ is all continuous functions on E which are holomorphic at interior points [5]. This yields a corresponding description of A.

Wermer's assumption that $M_A = M$ can be shown to be necessary, since the maximal ideal space of $R(E)$ is always E.

We consider a simple example in which $f(z) = |z|^2$. Here $E = \{0\}$ so that Wermer's conclusion would amount to $A = C(M)$. But f is constant on each circle C concentric with the origin, so each function in A is the uniform limit on C of polynomials in z. This property is clearly not possessed by every continuous function, so we do not have $A = C(M)$. In this case $F(M) = \{(z, w)\colon w = |z|^2 \leqq 1\}$, which is not polynomially convex since at each point of the form $(0, r^2)$ with $0 < r \leqq 1$ the maximum principle applied to a polynomial p considered as a function of the first variable shows that

$$|p(0, r^2)| \leqq \sup_{|z| = r} |p(z, r^2)|.$$

In fact, the polynomial hull of $F(M)$ is just its ordinary convex hull.

Wermer's conclusion is easily expressed as the conjunction of the two statements

(1) A contains the ideal of continuous functions which vanish on E;

(2) $R(E) = \{f | E \colon f \in A\}$.

His view suggests the consideration of two continuously differentiable functions f and g, and the set

$$E = \{\zeta \in M \colon df \wedge dg(\zeta) = 0\}.$$

This definition of E reduces to the one above if $g = z$, since $\bar{\partial} f = 0$ if and only if df is a multiple of dz.

In this more general situation statement (1) makes sense and is true [*1*]. However it is necessary to replace statement (2) by a description involving f and g. A possible one is suggested below. Of course, if either f or g is a diffeomorphism the problem is reduced to Wermer's case in an obvious manner. On the other hand, the absence of such a global coordinate causes severe difficulties.

But let us be bold and generalize considerably, to the case where M is a compact subset of a real continuously differentiable manifold of dimension n, F a set of continuously differentiable complex valued functions on M, and A the function algebra generated by F. Here we set

$$E = \{p \in M \colon df_1 \wedge \cdots \wedge df_n(p) = 0 \text{ for all } n\text{-tuples } f_1, \cdots, f_n \text{ of functions in } F\}.$$

Then statement (1) is true if $M_A = M$ and

(a) M is contained in a sufficiently differentiable manifold of arbitrary dimension embedded in $\mathbf{C}^n$, and $E = \varnothing$ (so that $A = C(M)$) [6], and

(b) M is contained in a real-analytic manifold of arbitrary dimension, and F is an arbitrary set of real-analytic functions [2].

As an example where statement (1) can easily be verified, let $M = \{(x,y,t) : x^2 + y^2 + t^2 \leqq 1\}$, $f(x,y,t) = x + iy = z$, $g(x,y,t) = t\bar{z}$, and $h(x,y,t) = t$. If $F = \{f,g,h\}$, it is easily seen that $F(M)$ is polynomially convex in $\mathbf{C}^3$, and also that $E = \{(x,y,t) : t = 0\}$. A contains the set of functions fh, g, and h, which is closed under complex conjugation, separates the points of M–E, and has no common zero there. An application of the Stone-Weierstrass theorem shows that A verifies statement (1).

To obtain a generalization of Wermer's conclusions, let us consider the set $H(F)$ of all continuously differentiable functions f for which at each point p in M there exist $f_1, \cdots, f_n$ in F and complex numbers $\lambda_1, \cdots, \lambda_n$ such that

$$df = \lambda_1 df_1 + \cdots + \lambda_n df_n,$$

where df denotes the differential of f at p. In Wermer's example and the one above, $H(F)$ consists of all functions which satisfy the Cauchy-Riemann equations on E. When F is finite it transports $H(F)$ over onto the set of continuously differentiable functions on $F(M)$ which satisfy the induced or "tangential" Cauchy-Riemann equations on $F(M)$.

It is clear that A is contained in the uniform closure $\overline{H(F)}$ of $H(F)$, since any polynomial in the functions of F is in $H(F)$. A standard conjecture is that if $M_A = M$, then A equals $\overline{H(F)}$. This is verified in Wermer's case [8] and in the three-dimensional example above. It is also true if M happens to be a polynomially convex real submanifold of $\mathbf{C}^n$ which is a Reinhardt set, meaning that together with each of its points z, M contains the orbit of z under the natural action of the n-torus on $\mathbf{C}^n$. It will be observed that these are very sharp restrictions on M, but the result is not completely trivial even for this case.

The proof, as pointed out by H. Rossi, is a straightforward extension to higher dimensions of a method which can be used for $n = 1$, when M is a closed disk. Stokes's theorem is used to show that the multivariate Fourier coefficients of a function f in $H(F)$ on different tori are related as if f were holomorphic in a neighborhood of M, and to show that these coefficients vanish at any multi-index which has a negative integer in it. These properties mean that there is a power series in $z_1, \cdots, z_n$ whose restriction to any torus T in M is the Fourier series for f on T. The uniform continuity of f on M

is used to show that the Cesaro means of this power series converge uniformly to f on M. This type of argument was used in $\mathbf{C}^n$ originally by K. de Leeuw [*4*].

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